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ISOMORPHIC SHEAF REPRESENTATIONS OF NORMAL LATTICES

George GEORGESCU and Ioana VOICULESCU
Institute of Mathematics, Bucharest, Romania

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For a distributive lattice with 0, 1 we obtain a sheaf representation which can be defined on any subframe of ideals of the given lattice. For the case of normal lattices and the subframe of σ-ideals, this sheaf is isomorphic to both sheaves induced on the space of maximal ideals by the Brezuleanu–Diaconescu [4] and Cornish [8] representations. As it turns out, the σ-ideals studied in [8], are the lattice-theoretic version of virginal ideals in rings and for normal lattices the situation parallels closely the one for commutative Gelfand rings [3, 12].

1.

This section contains some frame-theoretic results which are the abstract version of some facts which appear to be common to both contexts of commutative Gelfand rings and normal lattices respectively.

The background for this section can be found in [9].

A *frame* is a complete lattice A satisfying the infinite distributive law $a \wedge \bigvee S = \bigvee \{a \wedge s \mid s \in S\}$ for every $a \in A$, $S \subseteq A$ ($\wedge, \vee$ will denote binary meets and joins in A and $\bigvee$ denotes arbitrary joins).

A *frame morphism* is a function preserving finite meets and arbitrary joins. A *point* of a frame A is a prime element of A, i.e., an element generating a prime principal ideal. Points of A correspond bijectively to frame morphisms $\bar{p} : A \to \{0, 1\}$; the kernel of such a morphism must be a prime principal ideal whose generator is a point [9]. $\operatorname{Pt} A$ will denote the set of points of A.

With any $a \in A$ is associated a set of points:

$$d(a) = \{p \in \operatorname{Pt} A \mid a \not\leq p\}.$$

The sets $d(a)$ are the open sets of a topology on $\operatorname{Pt} A$. In what follows, $\operatorname{Pt} A$ will always be considered with this topology. The map d is a frame morphism from A onto the frame $\mathcal{O}(\operatorname{Pt} A)$ of open sets of $\operatorname{Pt} A$. The frame A is called *spatial*, or said to have *enough points* if d is an isomorphism.

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Let A be a frame and Φ a subframe of A. Let $p \in \operatorname{Pt} A$ and $\bar{p}: A \to \{0,1\}$ the frame morphism whose kernel is the prime principal ideal generated by p. The restriction $r\bar{p}$ of $\bar{p}$ to Φ has the kernel generated by $r(p) = \bigvee \{b \in \Phi \mid p(b) = 0\}$; $r(p)$ is thus a point of Φ and obviously it is the largest element in $\Phi, \leq p$.

Definition 1.1. If Φ is a subframe of the frame A, the map $r: A \to \Phi$ will be defined by

$$r(a) = \bigvee \{b \in \Phi \mid b \leq a\}, \quad a \in A.$$

r is the right adjoint of the inclusion $\Phi \to A$. By the preceding remarks, r maps any point of A onto a point of Φ.

If A is a frame and Φ is a subframe of A, we shall denote by $d(a)$, $a \in A$ and $t(b)$, $b \in \Phi$ the open sets in $\operatorname{Pt} A$ and $\operatorname{Pt} \Phi$ respectively.

Lemma 1.1. (i) *For any subframe Φ of a frame A, the map $r: \operatorname{Pt} A \to \operatorname{Pt} \Phi$ is continuous.*

(ii) *If A is spatial, any subframe Φ of A is spatial.*

Proof. (i) For $b \in \Phi$, $t(b) = \{p \in \operatorname{Pt} \Phi \mid b \not\leq p\}$. Then $a \in r^{-1}(t(b))$ iff $b \not\leq r(a)$ iff $b \not\leq a$, hence $r^{-1}(t(b)) = d(b)$ and r is continuous.

(ii) Let $b_1, b_2 \in \Phi$, $b_1 \neq b_2$. Since A is spatial, there exists $p \in \operatorname{Pt} A$ with $b_1 \not\leq p$, $b_2 \leq p$. But $r(p)$ is a point in Φ and $b_1 \not\leq r(p)$, $b_2 \leq r(p)$, hence $t(b_1) \neq t(b_2)$.

Proposition 1.2. *Let A be a spatial frame, Φ a subframe of A and C a subspace of* $\operatorname{Pt} A$ *such that:*

(*) $r: C \to \operatorname{Pt} \Phi$ *is surjective.*

(**) $r(a) \leq c \Rightarrow a \leq c$ *for any $a \in A$, $c \in C$.*

Then:

(i) *r is a homeomorphism between C and* $\operatorname{Pt} \Phi$.

(ii) *If $\mathscr{F}$ is any sheaf of algebraic structures on* $\operatorname{Pt} A$, *then its direct image along* $r: \operatorname{Pt} A \to \operatorname{Pt} \Phi$ *and the sheaf induced by $\mathscr{F}$ on C are isomorphic.*

Proof. (i) Let $a \in A$ and $\delta(a) = \{c \in C \mid a \not\leq c\}$ be an open subset in C. If $b \in \operatorname{Pt} \Phi$, then $b = r(c)$ for some $c \in C$ and by (*) $r(a) \not\leq r(c)$ iff $a \not\leq c$ iff $c \in \delta(a)$. Then $r(\delta(d)) = t(r(a))$ and $r|_C$ is an open map.

Let $c, d \in C$ with $r(c) = r(d)$. Then $r(d) \leq c$, $r(c) \leq d$ and by (**) it follows that $c = d$ and $r|_C$ is injective. Then by (*) and Lemma 1.1, r is a homeomorphism.

(ii) Let $\mathscr{F}'$ denote the sheaf induced on C by $\mathscr{F}$ and $r_*\mathscr{F}$ denote the direct image of $\mathscr{F}$ along r. By (i), the base spaces $\operatorname{Pt} \Phi$ and C of the two sheaves are homeomorphic. For $c \in C$ the stalks of $\mathscr{F}'$, $r_*\mathscr{F}$ at c and $r(c)$, respectively, are:

$$\mathscr{F}'_c = \mathscr{F}_c = \varinjlim_{\substack{a \in A \\ a \not\leq c}} \mathscr{F}(d(a)),$$

$$(r_*\mathscr{F})_{r(c)} = \varinjlim_{\substack{b\in\Phi \\ b\nleq r(c)}} \mathscr{F}(r^{-1}(t(b))) = \varinjlim_{\substack{b\in\Phi \\ b\nleq r(c)}} \mathscr{F}(d(b)).$$

For any $a\in A$, $a\nleq c \Rightarrow r(a)\nleq c \Rightarrow r(a)\nleq r(c)$, that is, the set $\{b\in\Phi \mid b\nleq r(c)\}$ is coinitial in $\{a\in A \mid a\nleq c\}$ and this implies $\mathscr{F}'_c \simeq (r_*\mathscr{F})_{r(c)}$.

Let E' and F be the display spaces of $\mathscr{F}'$ and $r_*\mathscr{F}$ respectively, and $f: E'\to F$ the map induced by the isomorphisms $f_c: \mathscr{F}'_c \to (r_*\mathscr{F})_{r(c)}$ between stalks.

It suffices to prove that f is continuous. If $c\in C$ and $x\in\mathscr{F}_c$, let $f(x,c) = (f_c(x), r(c))$. For a basic open set $\sigma(t(b))$ in F with σ a continuous section on $t(b)$, we have:

$$\begin{aligned} f^{-1}(\sigma(t(b))) &= \{(x,c) \mid r(c)\in t(b),\ \sigma(r(c)) = (f_c(x), r(c))\} \\ &= \{(x,c) \mid b\nleq r(c),\ \sigma(r(c)) = (f_c(x), r(c))\} \\ &= \{(x,c) \mid b\nleq c,\ \sigma(r(c)) = (f_c(x), r(c))\} \\ &= \{(x,c) \mid c\in\delta(b),\ \sigma(r(c)) = (f_c(x), r(c))\} \\ &= (f^{-1}\circ\sigma\circ r)(\delta(b)). \end{aligned}$$

From the definitions of $\mathscr{F}'$ and $r_*\mathscr{F}$ it follows that $f^{-1}\circ\sigma\circ r$ is a section in $\mathscr{F}'$, hence $f^{-1}(\sigma(t(b)))$ is open and f is continuous.

2.

Throughout this paper, lattice will mean distributive lattice with 0, 1 and Ld(0, 1) will denote the category of lattices and 0, 1 preserving morphisms between them.

Let L be any lattice in Ld(0, 1). If I is an ideal and F is a filter in L, L/I, L/F will denote the quotient of L with respect to the following congruences:

$$x \equiv_I y \quad \text{iff} \quad \text{there exists } a\in I \text{ with } x\vee a = y\vee a,$$

and respectively

$$x \equiv_F y \quad \text{iff} \quad \text{there exists } b\in F \text{ with } x\wedge b = y\wedge b.$$

If $x\in L$, $(x]$ and $[x)$ will denote the principal ideal, and principal filter respectively, generated by x. $x^\perp$ will denote the ideal $\{y\in L \mid x\wedge y = 0\}$. With any prime ideal P in L is associated [6] the ideal:

$$O(P) = \{x\in L \mid \text{there exists } y\in L-P,\ x\wedge y = 0\}.$$

Id L will denote the frame of ideals of L. Then Id L is spatial and Pt Id L = Spec L, the prime spectrum of L with the hull-kernel topology [9].

For any lattice L, two sheaf representations $\mathscr{L}$, $\mathscr{P}$ on Spec L were obtained by Brezuleanu–Diaconescu [4] and Cornish [8] respectively. The sheaf $\mathscr{L}$ associates to any basic open set $d((x])$ in Spec L, the quotient $L/[x)$, the restriction morphisms being the obvious ones. The stalks of $\mathscr{L}$ are the local lattices $L_P = L/(L-P)$, $P \in \operatorname{Spec} L$ [4]. The stalks of $\mathscr{P}$ are $L/O(P)$, $P \in \operatorname{Spec} L$. The basic open sets of $\coprod L/O(P)$ are $\{\hat{a}(d((x])) \mid a \in L, x \in L\}$ where $\hat{a}(P) = a/O(P)$, $P \in \operatorname{Spec} L$. In general, the lattices of sections of $\mathscr{P}$, over an arbitrary open set in $\operatorname{Spec} L$ are not known [8]. The lattices of all global sections for both $\mathscr{P}$ and $\mathscr{L}$ are isomorphic to L.

We shall describe a sheaf of lattices which can be defined on any subframe of $\operatorname{Id} L$, for a given lattice L and whose lattice of global sections is isomorphic to L.

Let us notice that by Lemma 1.1 and a usual argument we have:

Lemma 2.1. *If L is a lattice in* $\mathrm{Ld}(0,1)$, *any subframe of* $\operatorname{Id} L$ *is spatial and* $\operatorname{Pt} \Phi$ *is a compact T_0-space.*

If I is an ideal in L, an *I-multiplier* [7,10] is a map $f: I \to L$ such that $f(x \wedge y) = x \wedge f(y)$ for any $x \in L$, $y \in I$. Any multiplier preserves finite meets and joins and for any $x \in I$, $f(x) \leq x$ hence $f(x) \in I$.

We shall denote by $\mathscr{M}(I,L)$ the set of all I-multipliers. If the meet and join of two multipliers is defined pointwise, $\mathscr{M}(I,L)$ is a distributive lattice whose first element is the constant map 0 and the last element is the identity map on I (see [7,10]). If $a \in L$, the map f_a defined by $f_a(x) = a \wedge x$, $x \in L$ is an L-multiplier and this gives an isomorphism between L and $\mathscr{M}(L,L)$ (see [7]).

Let L be a lattice and Φ a subframe of $\operatorname{Id} L$. We define a presheaf Σ_Φ on $\Phi \simeq \mathscr{O}(\operatorname{Pt} \Phi)$ by: $\Sigma_\Phi(I) = \mathscr{M}(I,L)$ and for $I, J \in \Phi$ with $J \subseteq I$, the restriction map $\mathscr{M}(I,L) \to \mathscr{M}(J,L)$ maps any I-multiplier onto its restriction to J.

Theorem 2.2. *For any lattice in* $\mathrm{Ld}(0,1)$ *and any subframe Φ of* $\operatorname{Id} L$, *Σ_Φ is a sheaf whose lattice of global sections is isomorphic to L. $\Sigma_{\operatorname{Id} L}$ is isomorphic to $\mathscr{L}$ and Σ_Φ is isomorphic to the direct image of $\mathscr{L}$ along $r: \operatorname{Spec} L \to \operatorname{Pt} \Phi$.*

Proof. For $I \in \Phi$, suppose $t(I) = \bigcup \{t(I_\alpha) \mid \alpha \in U\}$ and let $f_\alpha \in (I_\alpha, L)$ be such that $f_\alpha|_{I_\alpha \cap I_\beta} = f_\beta|_{I_\alpha \cap I_\beta}$ for any $\alpha, \beta \in U$.

Then $I = \bigvee \{I_\alpha \mid \alpha \in U\}$ and for $a_i \in I_{\alpha_i}$, $i = 1, \ldots, n$, $b_j \in I_{\beta_j}$, $j = 1, \ldots, m$, the following implication holds:

$$(*) \qquad \bigvee_{i=1}^{n} a_i = \bigvee_{j=1}^{m} b_j \;\Rightarrow\; \bigvee_{i=1}^{n} f_{\alpha_i}(a_i) = \bigvee_{j=1}^{m} f_{\beta_j}(b_j).$$

Indeed, $a_i \wedge b_j \in I_{\alpha_i} \cap I_{\beta_j}$ and since $a_i \leq \bigvee_{j=1}^m b_j$, $1 \leq i \leq n$, $a_i = \bigvee_{j=1}^m (a_i \wedge b_j)$, hence:

$$\begin{aligned} f_{\alpha_i}(a_i) &= \bigvee_{j=1}^{m} f_{\alpha_i}(a_i \wedge b_j) = \bigvee_{j=1}^{m} f_{\beta_j}(a_i \wedge b_j) \\ &= \bigvee_{j=1}^{m} (a_i \wedge f_{\beta_j}(b_j)) \leq \bigvee_{j=1}^{m} f_{\beta_j}(b_j). \end{aligned}$$

It follows that $\bigvee_{i=1}^n f_{\alpha_i}(a_i) \leq \bigvee_{j=1}^m f_{\beta_j}(b_j)$ and conversely, which proves $(*)$. We can therefore define a function $f: I \to L$ by $f(a) = \bigvee_{i=1}^n f_{\alpha_i}(a_i)$, for any $a = \bigvee_{i=1}^n a_i$, with $a_i \in I_{\alpha_i}$, $i = 1, \dots, n$. f is an I-multiplier, for let $a = \bigvee_{i=1}^n a_i$, $a_i \in I_{\alpha_i}$, $i = 1, \dots, n$ and $b \in L$; then

$$f(a \wedge b) = f\left(\bigvee_{i=1}^n (a_i \wedge b)\right) = \bigvee_{i=1}^n f_{\alpha_i}(a_i \wedge b)$$

$$= \bigvee_{i=1}^n (f_{\alpha_i}(a_i) \wedge b) = f(a) \wedge b.$$

Now let $g \in \mathcal{M}(I, L)$ be such that $g|_{I_\alpha} = f_\alpha$, $\alpha \in U$. Then, if $a = \bigvee_{i=1}^n a_i$, $a_i \in I_{\alpha_i}$, $i = 1, \dots, n$:

$$g(a) = g\left(\bigvee_{i=1}^n a_i\right) = \bigvee_{i=1}^n g(a_i) = \bigvee_{i=1}^n f_{\alpha_i}(a_i) = f(a),$$

hence $f = g$ and Σ_Φ is a sheaf.

In order to prove the second assertion we show first that $\Sigma_{\mathrm{Id}\, L}$ is isomorphic to $\mathcal{L}$. Let $x \in L$ and define a map $\lambda_x: \mathcal{M}((x], L) \to L/[x)$ by $\lambda_x(f) = g(x)/[x)$. For $f, g \in \mathcal{M}((x], L)$ suppose $f(x)/[x) = g(x)/[x)$. Then $f(x) \wedge x = g(x) \wedge x$, hence $f(x) = g(x)$ since f, g are multipliers. But this implies easily that $f = g$, i.e., λ_x is injective. Let now $y/[x) \in L/[x)$ and $y_0 = x \wedge y$.

If $f_{y_0} \in \mathcal{M}((x], L)$ is given by $f_{y_0}(z) = z \wedge y_0$, $z \in (x]$, then

$$\lambda_x(f_{y_0}) = (y_0 \wedge x)/[x) = (x \wedge y)/[x) = y/[x)$$

and λ_x is surjective. Obviously λ_x is a lattice morphism, hence an isomorphism. It is functorial for let $y \le x$ and $f \in \mathcal{M}((x], L)$; then $f(x) \wedge y = f(x \wedge y) = f(y)$, $f(y) \wedge y = f(y)$, i.e., $f(x)/[y) = f(y)/[y)$ which shows that $f(x)/[y) = \lambda_y(f \mid (y])$. Since $d((x])$ are the basic open sets in $\operatorname{Spec} L$, the isomorphism of the two sheaves follows.

If Φ is any subframe of $\mathrm{Id}\, L$, $I \in \Phi$ and $r_* \mathcal{L}$ denotes the direct image of $\mathcal{L}$, we get:

$$r_* \mathcal{L}(t(I)) = \mathcal{L}(d(I)) \simeq \mathcal{M}(I, L) = \Sigma_\Phi(I),$$

hence $\Sigma_\Phi \simeq r_* \mathcal{L}$.

3.

Let L be any lattice in $\mathrm{Ld}(0,1)$. For $x \in L$, let us denote by $x^\perp$ the ideal $\{y \in L \mid x \wedge y = 0\}$. In [8], with any $I \in \mathrm{Id}\, L$ is associated the set

$$\sigma(I) = \{a \in L \mid I \vee a^\perp = L\}.$$

Then $\sigma(I)$ is an ideal and $\sigma(I) \subseteq I$. An ideal $I \in \mathrm{Id}\, L$ is called a *σ-ideal* if for any $a \in L$, $I \vee a^\perp = L$ (see [8]). Equivalently, I is a σ-ideal iff $I = \sigma(I)$.

Lemma 3.1 [8]. *The set of σ-ideals of a lattice L is a subframe of* $\mathrm{Id}\, L$.

We shall denote by ΦL the frame σ-ideals of a lattice L. By Theorem 2.2, we have:

Proposition 3.2. *Any lattice L in* Ld(0, 1) *is isomorphic to the lattice of global sections on a sheaf defined on the frame of σ-ideals in L.*

If $r: \mathrm{Id}\, L \to \Phi L$ is the map in Definition 1.1, it is immediate that

$$r(I) \subseteq \sigma(I) \subseteq I \quad \text{for any ideal } I \text{ in } L.$$

Let M be a maximal ideal in L. Then $M \vee x^{\perp} = L$ iff $x^{\perp} \not\subseteq M$ iff $x \in O(M)$, hence:

$$\sigma(M) = O(M) \quad \text{for any maximal ideal } M.$$

A lattice L is called *normal* if for any $x, y \in L$ with $x \vee y = 1$ there exist $x_1, y_1 \in L$ such that $x \vee x_1 = 1$, $y \vee y_1 = 1$ and $x_1 \wedge y_1 = 0$. Denote by $\mathrm{Max}\, L$ the set of maximal ideals of a lattice L.

Lemma 3.2 ([9, 11]). *For a lattice L the following are equivalent:*

(i) *L is normal.*

(ii) *Any prime ideal of L is contained in a unique maximal ideal.*

(iii) *The inclusion $i: \mathrm{Max}\, L \to \mathrm{Spec}\, L$ has a continuous retraction.*

(iv) *For any pair of distinct maximal ideals M, N of L, there exist $x \in M - N$, $y \in N - M$ with $x \wedge y = 0$.*

Lemma 3.3. *Let L be a normal lattice, $I \in \mathrm{Id}\, L$, $M \in \mathrm{Max}\, L$.*

(1) $\sigma(M) \subseteq I \Rightarrow I \subseteq M$ *or* $I = L$.

(2) $\sigma(I) \subseteq M \Rightarrow I \subseteq M$.

(3) $\sigma(I) = r(I)$.

(4) $r(I) \subseteq M \Leftrightarrow I \subseteq M$.

Proof. (1) If $I \neq L$, let $N \in \mathrm{Max}\, L$, $I \subseteq N$. If $M \neq N$, let $a \in M - N$, $b \in N - M$ with $a \wedge b = 0$. Then $M \vee (b] = L$ and there exists $x \in M$, $x \vee b = 1$; this implies $M \vee a^{\perp} = L$, i.e., $a \in \sigma(M) \subseteq I \subseteq N$, which contradicts $a \in M - N$. Hence $M = N$ and $I \subseteq M$.

(2) Dualizing an argument in [6], M is the unique maximal ideal which contains $O(M)$. Suppose $\sigma(I) \subseteq M$ and $I \not\subseteq M$. Then by (1), $\sigma(M) = O(M) \not\subseteq I$ and $\sigma(M) \vee I = L$ since otherwise $\sigma(M) \vee I \subseteq M$, hence $I \subseteq M$. Let $x \in \sigma(M)$, $y \in I$ such that $x \vee y = 1$; $x \in \sigma(M)$ implies there exist $m \in M$, $u \in L$ with $m \vee u = 1$, $x \wedge u = 0$. Hence $I \vee u^{\perp} = L$, $u \in \sigma(I) \subseteq M$ and $m \vee u = 1$, which contradicts M being a proper ideal.

(3) Let $x \in \sigma(I)$ and suppose $\sigma(I) \vee x^{\perp} \neq L$. Then let N be a maximal ideal containing $\sigma(I) \vee x^{\perp}$. From (2), $I \subseteq N$, hence $I \vee x^{\perp} = L \subseteq N$. It follows that $\sigma(I)$ is a σ-ideal and since $r(I)$ is the largest σ-ideal contained in I, $\sigma(I) = r(I)$.

(4) $r(I) \subseteq M \Rightarrow I \subseteq M$ follows from (2), (3) and the converse implication is trivial.

Using the above lemma, we obtain a characterization of normal lattices by means of the map r into the frame of σ-ideals.

Proposition 3.4. *For a lattice L, the following are equivalent:*
(a) $I \vee J = L \Rightarrow r(I) \vee r(J) = L$, *for any* $I, J \in \operatorname{Id} L$.
(b) L *is normal.*
(c) $r : \operatorname{Id} L \to \Phi L$ *preserves arbitrary joins.*

Proof. (a) $\Rightarrow$ (b). Since L is normal iff $\operatorname{Id} L$ is normal, we shall prove this last property. Let $I, J \in \operatorname{Id} L$ with $I \vee J = L$. Then $r(I) \vee r(J) = L$ and let $a \in r(I)$, $b \in r(J)$ with $a \vee b = 1$. Then $r(I) \vee a^{\perp} = L$, hence $r(I) \vee r(a^{\perp}) = L$, and let $x \in r(I)$, $y \in r(a^{\perp})$ with $x \vee y = 1$. Then $I \vee x^{\perp} = L$, $J \vee b^{\perp} = L$ and if $x^{\perp} \cap b^{\perp} = 0$ it follows that $\operatorname{Id} L$, hence L is normal. Let $c \in x^{\perp} \cap b^{\perp}$, i.e., $c \wedge x = 0$, $c \wedge b = 0$. Then $c = c \wedge (a \vee b) = c \wedge a$, $c = c \wedge (x \vee y) = c \wedge y$, that is $c \leq a \wedge y = 0$.

(b) $\Rightarrow$ (c). Let $I_t \in \operatorname{Id} L$, $t \in T$. Then for any lattice L

$$\bigvee \{r(I_t) \mid t \in T\} \subseteq r(\bigvee \{I_t \mid t \in T\}).$$

For the converse inclusion, let $x \in r(\bigvee \{I_t \mid t \in T\})$. Let $\bigvee \{I_t \mid t \in T\} \vee x^{\perp} = L$, but suppose $K = \bigvee \{r(I_t) \mid t \in T\} \vee x^{\perp} \neq L$. Let M be a maximal ideal containing K; for any $t \in T$, $r(I_t) \subseteq M$, and by Lemma 3.3, $I_t \subseteq M$. Hence $\bigvee \{I_t \mid t \in T\} \vee x^{\perp} \subseteq M$, which is contradictory. It follows that $\bigvee \{I_t \mid t \in T\} \vee x^{\perp} = L$, hence $x \in \bigvee \{r(I_t) \mid t \in T\}$.

(c) $\Rightarrow$ (a). Obvious.

Lemma 3.5. *If L is a normal lattice, then* $\operatorname{Max} L$ *and* $\operatorname{Pt} \Phi L$ are homeomorphic.

Proof. By Proposition 1.2 and Lemma 3.3, with $A = \operatorname{Id} L$, $C = \operatorname{Max} L$, $\Phi = \Phi L$ it suffices to prove that $r : \operatorname{Max} L \to \operatorname{Pt} \Phi L$ is surjective.

Let $P \in \operatorname{Pt} \Phi L$ and M be a maximal ideal containing P. We show that $r(M) \subseteq P$, hence $P = r(M)$. Let $a \in r(M)$; $r(M) \vee a^{\perp} = L$ hence there exist $x \in r(M)$, $y \in a^{\perp}$ with $x \vee y = 1$. $x \in r(M)$ implies, by Proposition 3.4, $r(M) \vee r(x^{\perp}) = L$, hence $r(x^{\perp}) \not\subseteq M$ and therefore $r(x^{\perp}) \not\subseteq P$.

Obviously $r((x]) \cap r(x^{\perp}) = \{0\}$ and since P is a point in ΦL, and $r(x^{\perp}) \not\subseteq P$, $r((x]) \subseteq P$. But $(x] \vee a^{\perp} = L$, that is $a \in \sigma((x]) = r((x]) \subseteq P$, and $r(M) \subseteq P$.

In a normal lattice, $\operatorname{Max} L$ is Hausdorff [9], hence $\operatorname{Pt} \Phi L$ has the same property. The following gives a converse to the above lemma.

Lemma 3.6. *Let Φ be some subframe of* $\operatorname{Id} L$.
(a) *If* $\operatorname{Pt} \Phi$ *is* T_1, *then* $r : \operatorname{Max} L \to \operatorname{Pt} \Phi$ *is surjective.*
(b) *If* $\operatorname{Pt} \Phi$ *is* T_2 *and r is injective on* $\operatorname{Max} L$, *then L is normal.*

Proof. (a) Let us notice that if $\operatorname{Pt} \Phi$ is a T_1-space, any point in Φ is a maximal element in $\Phi - \{L\}$. Let $P \in \operatorname{Pt} \Phi$ and M be a maximal ideal containing P. Then $P \subseteq r(M)$, $r(M) \in \operatorname{Pt} \Phi$ and since P is maximal in $\Phi - \{L\}$, $P = r(M)$.

(b) Let Q a prime ideal of L and suppose M_1, M_2 are maximal ideals containing Q. Then $r(Q) \subseteq r(M_1)$, $r(Q) \subseteq r(M_2)$, $r(Q)$ is a point of Φ, hence a maximal element and therefore $r(Q)=r(M_1)=r(M_2)$, that is, $M_1=M_2$ by injectivity of r. It follows that L is normal.

Corollary 3.7. *The lattice L is normal iff $r:\operatorname{Max} L\to \operatorname{Pt} \Phi L$ is a homeomorphism.*

Lemma 3.8. *If M is a maximal ideal in the normal lattice L, then L_M and $L/O(M)$ are isomorphic.*

Proof. Suppose $x\equiv_{L-M} y$; then $x\wedge t=y\wedge t$ for some $t\in L-M$ and since M is maximal, $M\vee(t]=L$, hence $1=m\vee t$ for some $m\in M$. L is normal, so there exist $m_1, t_1\in L$ with $m_1\vee m=1$, $t_1\vee t=1$, $m_1\wedge t_1=0$. Then $t_1\in O(M)$, $x\vee t_1=y\vee t_1$, hence $x\equiv_{O(M)} y$. The converse implication holds in any lattice and its proof is immediate. It follows that $L_M, L/O(M)$ are isomorphic.

Theorem 3.9. *Let L be a normal lattice. Then $\Sigma_{\Phi L}$ is isomorphic to the sheaves induced on $\operatorname{Max} L$ by $\mathscr{L}$ and $\mathscr{P}$.*

Proof. The isomorphism of $\Sigma_{\Phi L}$ to the sheaf induced on $\operatorname{Max} L$ *by* $\mathscr{L}$ follows from Proposition 1.2, Theorem 2.2, and Lemma 3.5.

Let $\mathscr{P}'$ denote the sheaf induced on $\operatorname{Max} L$ by $\mathscr{P}$. By Lemma 3.8, $\Sigma_{\Phi L}$ and $\mathscr{P}'$ have isomorphic stalks; the isomorphism of the two sheaves follows easily.

4.

Let R be a commutative ring with unit. The *reticulation* of R, defined in [11] (see also [9]), is the lattice LR generated by symbols $D(a)$, $a\in R$, and satisfying the conditions

$$D(1_R)=1_{LR},\qquad D(0_R)=0_{LR},$$

$$D(ab)=D(a)\wedge D(b),\qquad D(a+b)\le D(a)+D(b).$$

For an ideal I in R, $D[I]$ will denote the ideal in LR generated by $\{D(a)\mid a\in I\}$. If J is any ideal in LR, $D^{-1}(J)=\{a\in R\mid D(a)\in J\}$ is always an ideal of R. There is a frame isomorphism between the frame of radical ideals of R (denoted $\mathscr{R}\operatorname{Id} R$) and the frame of ideals of LR, $D:\mathscr{R}\operatorname{Id} R\to \operatorname{Id} LR$, given by $D(K)=D[K]$, $K\in \mathscr{R}\operatorname{Id} R$ (see [9, p. 194]).

Following the terminology in [3], an ideal I of R is called *virginal* if $I+\operatorname{Ann}(a)=R$ for any $a\in I$ ($\operatorname{Ann}(a)$ is the annihilator of a). The set ΦR of virginal ideals of R is a frame under the operations of finite intersection and arbitrary sum of ideals [3]. If I is any ideal in R, $\operatorname{Vir} I$ will denote the largest virginal ideal contained in I and $\operatorname{Ker} I$ will denote the ideal $\{a\in I\mid I+\operatorname{Ann}(a)\}=R$. Obviously, $\operatorname{Vir} I\subseteq \operatorname{Ker} I\subseteq I$.

As remarked in [9], the function which maps any ideal I of R onto its radical $\sqrt{I}$

is a frame morphism from ΦR to $\mathscr{R}\,\mathrm{Id}\,R$; its composition with D gives a frame morphism $D': R \xrightarrow{\sqrt{}} \mathscr{R}\,\mathrm{Id}\,R \to \mathrm{Id}\,LR$ from the frame of virginal ideals of R to $\mathrm{Id}\,LR$.

Proposition 4.1. *For any commutative ring with unit, R, the map $D': \Phi R \to \Phi LR$ is an isomorphism between the frame of virginal ideals of R and the frame of σ-ideals of the reticulation of R.*

Proof. We first prove that $\sqrt{\ }$ is injective on ΦR. Indeed, let $I_1, I_2 \in \Phi R$ and suppose $\sqrt{I_1} = \sqrt{I_2}$. If $\mathscr{P}_i, \mathscr{M}_i$ denote the sets of prime ideals and maximal ideals, respectively, above I_i, $i=1,2$, by hypothesis and by Lemma 2.5 in [3], the following equalities hold:

$$\bigcap \mathscr{P}_1 = \bigcap \mathscr{P}_2,$$

$$I_i = \mathrm{Vir} \bigcap \mathscr{M}_i, \quad i=1,2.$$

Since $I_i \subseteq \bigcap \mathscr{P}_i \subseteq \bigcap \mathscr{M}_i$, $i=1,2$, it follows by the monotony of Vir, that

$$I_i = \mathrm{Vir}\, I_i \subseteq \mathrm{Vir} \bigcap \mathscr{P}_i \subseteq \mathrm{Vir} \bigcap \mathscr{M}_i = I_i, \quad i=1,2.$$

hence $I_1 = \mathrm{Vir} \bigcap \mathscr{P}_1 = \mathrm{Vir} \bigcap \mathscr{P}_2 = I_2$ and the injectivity of $\sqrt{\ }$ on ΦR follows.

Let $I \in \Phi R$ and $x \in D[I] = D[\sqrt{I}] = D'(I)$. It is immediate to see that $x = D(a_1) \vee \cdots \vee D(a_n)$ for some $a_1, \ldots, a_n \in I$; since I is virginal:

$$I + \mathrm{Ann}(a_i) = R, \quad i=1, \ldots, n,$$

hence there exist $b_i \in I$, $c_i \in R$ such that

$$a_i + b_i = 1_R, \quad a_i c_i = 0_R, \qquad i=1, \ldots, n.$$

Then $(c_1+b_1) \cdots (c_n+b_n) = c_1 c_2 \cdots c_n + b = 1_R$, where $b \in I$ and $c = c_1 c_2 \cdots c_n \in \mathrm{Ann}(a_i)$, $i=1, \ldots, n$. It follows that:

$$D(b) \in D'(I), \qquad D(a_i) \wedge D(c) = 0_{LR}, \quad i=1, \ldots, n,$$

$$\begin{aligned} x \wedge D(c) &= (D(a_1) \vee \cdots \vee D(c)) \wedge D(c) \\ &= D(a_1 c) \vee \cdots \vee D(a_n c) = 0_{LR}, \end{aligned}$$

$$D(b) \vee D(c) \geq D(b+c) = 1_{LR},$$

i.e., $D'(I) \vee x^{\perp} = LR$ and $D'(I)$ is a σ-ideal of LR, for any virginal ideal I of R.

Conversely, let J be a σ-ideal of LR and $a \in D^{-1}(J)$. Then $D(a) \in J$, $J \vee D(a)^{\perp} = LR$, hence

$$D^{-1}(J) + D^{-1}(D(a)^{\perp}) = R.$$

Therefore there exist $b \in D^{-1}(J)$, $c \in D^{-1}(D(a)^{\perp})$, with $b + c = 1_R$, $D(a) \wedge D(c) = D(ac) = 0_{LR}$. By Lemma V.3.2 in [9], the last equality implies $a^n c^n = 0_R$ for some $n \in \mathbb{N}$ and we have:

$$1 = (b+c)^n = c^n + b', \quad b' \in D^{-1}(J),\ c^n \in \mathrm{Ann}(a^n), \text{ for some } n \in \mathbb{N},$$

i.e., $a^n \in \operatorname{Ker} D^{-1}(J)$ and $a \in \sqrt{\operatorname{Ker} D^{-1}(J)}$. We got the inclusion $D^{-1}(J) \subseteq \sqrt{\operatorname{Ker} D^{-1}(J)}$, which implies, since $D^{-1}(J)$ is a radical ideal, that:

$$(\alpha) \qquad D^{-1}(J) = \sqrt{\operatorname{Ker} D^{-1}(J)}, \quad J \in \Phi LR.$$

In order to show $\operatorname{Ker} D^{-1}(J) = \operatorname{Vir} D^{-1}(J)$, let $a \in \operatorname{Ker} D^{-1}(J)$. Then $D^{-1}(J) + \operatorname{Ann}(a) = R$ and, by (α), $\sqrt{\operatorname{Ker} D^{-1}(J)} + \operatorname{Ann}(a) = R$, i.e., there exist $b, c \in R$ such that $b + c = 1_R$, $c \in \operatorname{Ann}(a)$ and $b^n \in \operatorname{Ker} D^{-1}(J)$ for some $n \in \mathbb{N}$.

Hence $(b+c)^n = b^n + cd = 1_R$, $cd \in \operatorname{Ann}(a)$, that is, $\operatorname{Ker} D^{-1}(J) + \operatorname{Ann}(a) = R$ and:

$$(\beta) \qquad D^{-1}(J) = \sqrt{\operatorname{Vir} D^{-1}(J)}, \quad J \in \Phi LR,$$

$$(\gamma) \qquad J = D(\sqrt{\operatorname{Vir} D^{-1}(J)}) = D'(\operatorname{Vir} D^{-1}(J)), \quad J \in \Phi LR,$$

i.e., any σ-ideal of LR is the image of a virginal ideal of R, via D'.

It follows that $D' = D \circ \sqrt{\ }$ is a frame isomorphism between virginal ideals of a ring R and σ-ideals of its reticulation.

Corollary 4.2. *The space of prime virginal ideals of a commutative ring with unit is homeomorphic to the space of prime σ-ideals of its reticulation.*

Corollary 4.3. *For any ideal I of R and any ideal J of LR the following relations hold:*

(i) $D'(\operatorname{Vir} I) = r(D[I])$,

(ii) $D'^{-1}(r(J)) = \operatorname{Vir} D^{-1}(J)$.

Proof. (i) Since $D[I] = D(\sqrt{I})$, it is sufficient to prove (i) for radical ideals only. If I is a radical ideal, $\sqrt{\operatorname{Vir} I} \subseteq I$ and this implies

$$D'(\operatorname{Vir} I) = D(\sqrt{\operatorname{Vir} I}) \subseteq D(I), \qquad D'(\operatorname{Vir} I) \subseteq r(D(I)).$$

For the converse direction, $r(D(I)) \subseteq D(I)$ implies

$$D^{-1}(r(D(I))) \subseteq I, \qquad \operatorname{Vir} D^{-1}(r(D(I))) \subseteq \operatorname{Vir} I,$$

hence by (β),

$$D^{-1}(r(D(I))) \subseteq \sqrt{\operatorname{Vir} I} \quad \text{and} \quad r(D(I)) \subseteq D(\sqrt{\operatorname{Vir} I}) = D'(\operatorname{Vir} I).$$

(ii) follows from (i).

The preceding results show that, like the spectrum, the space of virginal ideals of a commutative ring R may be constructed as a space of ideals of its associated lattice LR. It also follows, by Corollary 4.3, that the results concerning the role of virginal ideals in commutative Gelfand rings [3] and of σ-ideals in normal lattices, respectively, can be translated without difficulty into each other via the reticulation map D.

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